Correction Model Linear Algebra 2
April 7, 2020

1 a) $A \in S$ if and only if $A^T = A$
Suppose $A, B \in S$ and $a, b \in \mathbb{R}$. To show
2) that $aA + bB \in S$. This is easy:

$$(aA + bB)^T = aA^T + bB^T = aA + bB.$$

b) I just take a try: a general element in S has the form $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} =$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ therefore span S. They are also linearly independent. Thus they form a <u>basis</u> of S. I now need to find an <u>orthonormal</u> basis.
Note:

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = 0$$

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = 0$$

$$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = 0$$

Hence they are <u>orthogonal</u>. Now compute their norms:

$$\left\langle \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix} \right\rangle = 1 \Rightarrow \left\| \begin{pmatrix}1&0\\0&0\end{pmatrix} \right\| = 1$$

$$\left\langle \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix} \right\rangle = 2 \Rightarrow \left\| \begin{pmatrix}0&1\\1&0\end{pmatrix} \right\| = \sqrt{2}$$

$$\left\langle \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\rangle = 1 \Rightarrow \left\| \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\| = 1$$

Conclusion :

7 . $$\left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$$

is an orthonormal basis.

c) Call the orthogonal projection $P \in S$
Then

$$\begin{aligned} P = & \left\langle \begin{pmatrix}a&b\\c&d\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix} \right\rangle \cdot \begin{pmatrix}1&0\\0&0\end{pmatrix} \\ & + \left\langle \begin{pmatrix}a&b\\c&d\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ & + \left\langle \begin{pmatrix}a&b\\c&d\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\rangle \cdot \begin{pmatrix}0&0\\0&1\end{pmatrix} \\ = & \, a\begin{pmatrix}1&0\\0&0\end{pmatrix} + \frac{1}{\sqrt{2}}(b+c)\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ & + d\begin{pmatrix}0&0\\0&1\end{pmatrix} \end{aligned}$$

7 $$= \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix}.$$

(2) a) Let $v_1, \dots v_r \in \mathbb{R}^n$ be the columns of V so

$$V = (v_1 \ v_2 \ v_3 \ \dots \ v_r)$$

Then $\{v_1, v_2, \dots v_r\}$ is a basis of the subspace $\mathcal{V}$. It is given that $\mathcal{V}$ is A-invariant. Therefore, for $i = 1, 2, 3, \dots, r$

$$A v_i \in \mathcal{V} = \text{span}(v_1, v_2, \dots v_r)$$

In other words, Av_i is a linear combination of $v_1, v_2, \dots v_r$. Thus there exist $a_{ij} \in \mathbb{R}$ such that

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$$\begin{aligned} A v_1 &= a_{11} v_1 + a_{21} v_2 + \dots + a_{r1} v_r \\ A v_2 &= a_{12} v_1 + a_{22} v_2 + \dots + a_{r2} v_r \\ &\vdots \\ A v_r &= a_{1r} v_1 + a_{2r} v_2 + \dots + a_{rr} v_r \end{aligned}$$

Define $A_{11} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & & a_{rr} \end{pmatrix}$

Then $AV = VA_{11}$

b) Let λ be an eigenvalue of A_{11}. Then there exists $x \neq 0$ such that $A_{11} x = \lambda x$

This implies

$$A V x = V A_{11} x = \lambda V x$$

Define $v := Vx$. Then $v \neq 0$ because $v = 0$ would imply $Vx = 0$, which would imply $x = 0$ (V has linearly independent columns). So: $v \neq 0$ and

(4)

$$A v = \lambda v$$

so λ is an eigenvalue of A.

c) We already saw that $\{v_1, v_2 \ldots v_r\}$ is a basis of $\mathcal{V}$. Extend this to a basis

$$\beta := \{v_1, v_2, \ldots, v_r, v_{r+1}, v_{r+2} \ldots v_n\}$$

of $\mathbb{R}^n$. Then the matrix of A with respect to β has the form

(6).

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} is the matrix already found in part a).

d) $$p_A(z) = \det \begin{pmatrix} A_{11} - zI & A_{12} \\ 0 & A_{22} - zI \end{pmatrix}$$

So

(4) $$p_A(z) = \det(A_{11} - zI)\det(A_{22} - zI)$$

$$= p_{A_{11}}(z)\, g(z)$$

where $g(z)$ is the polynomial $\det(A_{22} - zI)$. We conclude that $p_{A_{11}}(z)$ divides $p_A(z)$.

(3) a) I and A are simultaneously diagonalizable: it is given that A is diagonalizable so there exists nonsingular S such that

(3) $$S^{-1} A S = \Lambda \quad \text{is diagonal.}$$

Obviously

$$S^{-1} I S = I \quad \text{is diagonal}$$

so S diagonalizes both A and I.

b) There exist nonsingular S such that

$$S^{-1} A S = \Lambda \quad \text{is diagonal}$$
$$S^{-1} B S = M \quad \text{is diagonal.}$$

Then $A = S \Lambda S^{-1}$ and $B = S M S^{-1}$ so $AB = S \Lambda S^{-1} S M S^{-1} = S \Lambda M S^{-1} =$

Now $\Lambda M = M \Lambda$ because diagonal
5 matrices commute. This leads to

$$AB = SM\Lambda S^{-1} = SMS^{-1}S\Lambda S^{-1}$$
$$= BA.$$

c) Let $D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$ with $d_i \neq d_j$ $(i \neq j)$

Take any matrix $A = (a_{ij})$ such that $DA = AD$. Then

$$\begin{pmatrix} d_1 a_{11} & d_1 a_{12} & \cdots & d_n a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \cdots & d_2 a_{2n} \\ \vdots & \vdots & & \\ d_n a_{n1} & d_n a_{n2} & \cdots & d_n a_{nn} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} d_1 & a_{12} d_2 & \cdots & a_{1n} d_n \\ a_{21} d_1 & a_{22} d_2 & \cdots & a_{2n} d_n \\ \vdots & \vdots & & \vdots \\ a_{n1} d_1 & a_{n2} d_2 & \cdots & a_{nn} d_n \end{pmatrix}$$

By taking a careful look at this, we ~~see~~ that this holds iff $d_i a_{ij} = d_j a_{ij}$ for all $i \neq j$ so iff $(d_i - d_j) a_{ij}$ $(i \neq j)$, equivalently:

5 $a_{ij} = 0$ for all $i \neq j$ (since $d_i \neq d_j$)

So: $AD = DA \iff A$ is a diagonal matrix

d) Suppose $AB = BA$ and A has n distinct eigenvalues. In the first place we have that there exists nonsingular S such that

$$S^{-1} A S = D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix}$$

with $d_i \neq d_j$, $i \neq j$

Now consider $S^{-1} B S$ and call this matrix M. I will show that M is a diagonal matrix!

Indeed
$$\begin{aligned} DM &= SAS^{-1} \cdot SBS^{-1} \\ &= SABS^{-1} \\ &= SBAS^{-1} \\ &= SBS^{-1} \cdot SAS^{-1} \\ &= MD \end{aligned}$$

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Thus M must be diagonal by part c) of this problem.

4 a) ($\Rightarrow$) Assume $A>0$. To prove that $B^TAB>0$. Let $x\neq 0$ then $Bx\neq 0$ since B nonsingular. Thus

$$x^TB^TABx = (Bx)^TA(Bx)>0$$

($\Leftarrow$) Assume $B^TAB>0$. To prove $A>0$. Let $x\neq 0$. Then $B^{-1}x\neq 0$. Thus

4 we get

$$x^TAx = x^T(B^T)^{-1}B^TABB^{-1}x$$
$$= (B^{-1}x)^TB^TAB(B^{-1}x)>0$$

b) $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22}\end{pmatrix}$, A_{11}, A_{22} symmetric

First note that $A^T=A$ so symmetric

4 Assume $A>0$. Then $x^TAx>0$ for all $x\neq 0$. Now let $x_1\neq 0$ and take

$x=\begin{pmatrix} x_1 \\ 0\end{pmatrix}$. Then $x\neq 0$ and

$$x_1^TA_{11}x_1 = x^TAx>0 \quad \text{so } A_{11}>0$$

Likewise, let $x_2\neq 0$ and take $x=\begin{pmatrix} 0 \\ x_2\end{pmatrix}$

Then $x\neq 0$ and

$$x_2^TA_{22}x_2 = x^TAx>0 \quad \text{so } A_{22}>0$$

c)

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{12}^TA_{11}^{-1}A_{12} \end{pmatrix}\begin{pmatrix} I & X^T \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & 0 \\ XA_{11} & A_{22}-A_{12}^TA_{11}^{-1}A_{12} \end{pmatrix}\begin{pmatrix} I & X^T \\ 0 & I \end{pmatrix}$$

5 $$= \begin{pmatrix} A_{11} & A_{11}X^T \\ XA_{11} & XA_{11}X^T + A_{22}-A_{12}^TA_{11}^{-1}A_{12} \end{pmatrix}$$

In order for this to equal $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$

we should take $X = A_{12}^T A_{11}^{-1}$

d) ($\Rightarrow$) Assume $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} > 0$. Then

$A_{11} > 0$ [by part b)]. Thus A_{11} is nonsingular.

and

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{12}^TA_{11}^{-1}A_{12} \end{pmatrix}$$
$$\begin{pmatrix} I & X^T \\ 0 & I \end{pmatrix}$$

Since $\begin{pmatrix} I & X^T \\ 0 & I \end{pmatrix}$ is nonsingular this implies [by part a)] that

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{12}^T A_{11}^{-1} A_{12} \end{pmatrix} > 0$$

By part b) this yields $A_{11} > 0$ and $A_{22} - A_{12}^T A_{11}^{-1} A_{12} > 0$

($\Leftarrow$) If $A_{11} > 0$ and $A_{22} - A_{12}^T A_{11}^{-1} A_{12} > 0$ then $\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{12}^T A_{11}^{-1} A_{12} \end{pmatrix} > 0$

This implies $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{12}^T A_{11}^{-1} A_{12} \end{pmatrix} \begin{pmatrix} I & X^T \\ 0 & I \end{pmatrix} > 0$

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So $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} > 0$.

5 $\quad M = \begin{pmatrix} a & -b & -c \\ a & -b & c \\ a & b & -c \\ a & b & c \end{pmatrix} \qquad a > b > c.$

a) $\quad M^T M = \begin{pmatrix} 4a^2 & 0 & 0 \\ 0 & 4b^2 & 0 \\ 0 & 0 & 4c^2 \end{pmatrix}$

This matrix has eigenvalues $4a^2 > 4b^2 > 4c^2$
So the singular values of M are
$\sigma_1 = 2a$, $\sigma_2 = 2b$, $\sigma_3 = 2c$ with,
as required $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and $\sigma_3 > 0$

I compute orthonormal set of eigenvectors of $M^T M$

$$M^T M v_1 = 4a^2 v_1$$

$$M^T M v_2 = 4b^2 v_2$$

$$M^T M v_3 = 4c^2 v_3 .$$

So take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So: our "V" matrix is I_3, the 3×3 identity matrix.

We now compute $U = (u_1\ u_2\ u_3\ u_4)$.
We should have

$$M = U \Sigma V^T$$

So $\begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix} = 2a . u_1$, $\begin{pmatrix} -b \\ -b \\ b \\ b \end{pmatrix} = 2b . u_2$

and $\begin{pmatrix} -c \\ c \\ -c \\ c \end{pmatrix} = 2c\, u_3$

Thus : $u_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $u_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$u_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ (note : orthonormal !)

For u_4 we can take any vector such that $U = (u_1\ u_2\ u_3\ u_4)$ is an orthogonal matrix.

Take $u_4 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

So $U = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

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and the SVD is $M = U \begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \\ 0 & 0 & 0 \end{pmatrix} V^T$

with $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b) Best rank 2 approximation is given by

4 $$U\begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

6 A is a complex 5×5 matrix

a) Distinct eigenvalues

λ_1	$g_1=2$	$a_1=2$
λ_2	$g_2=1$	$a_2=2$
λ_3	$g_3=1$	$a_3=1$

- λ_1 has $g_1=2$ blocks, both of size 1×1 since $a_1=2$
- λ_2 has $g_2=1$ block, of size 2×2 since $a_2=2$
- λ_3 has $g_3=1$ block of size 1×1 since $a_3=1$.

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so $$J=\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

b) In general $p_{min}(z)=(z-\lambda_1)^{m_1}(z-\lambda_2)^{m_2}(z-\lambda_3)^{m_3}$
where m_1, m_2, m_3 are the maximal sizes

of the Jordan blocks. These are 1, 2, 1 respectively, as can seen from J.
Hence

(5) $$P_{min}(z) = (z-\lambda_1)(z-\lambda_2)^2(z-\lambda_3)$$

c) $A \in \mathbb{C}^{5\times 5}$ has two distinct eigenvalues λ_1, λ_2. $P_{min}(z) = (z-\lambda_1)(z-\lambda_2)^2$
For λ_1 we have $a_1 = 1$.
What do we know?

1. In J there is one 1×1 block with eigenvalues λ_1 since $a_1 = 1$
2. The largest block for eigenvalue λ_2 is equal to 2 (from $P_{min}(z)$).

This is all we know. This leads to the following possibilities for J:

$$J = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & 1 & & \\ & 0 & \lambda_2 & & \\ & & & \lambda_2 & 1 \\ & & & 0 & \lambda_2 \end{pmatrix}, \quad J = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_2 & & \\ & & & \lambda_2 & 1 \\ & & & 0 & \lambda_2 \end{pmatrix}$$ (8)

(or permutation of these blocks of course)